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Note on New KLT relations

Bo Feng[a,d], **Song He**[b,d], **Rijun Huang**[c,d], **Yin Jia**[c,d]

[a] *Center of Mathematical Science, Zhejiang University, Hangzhou, China*
[b] *Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Golm, Germany*
[c] *Zhejiang Institute of Modern Physics, Physics Department, Zhejiang University, Hangzhou, China*
[d] *Kavli Institute for Theoretical Physics China, CAS, Beijing 100190, China*

Abstract: In this short note, we present two results about KLT relations discussed in recent several papers. Our first result is the re-derivation of Mason-Skinner MHV amplitude by applying the S_{n-3} permutation symmetric KLT relations directly to MHV amplitude. Our second result is the equivalence proof of the newly discovered S_{n-2} permutation symmetric KLT relations and the well-known S_{n-3} permutation symmetric KLT relations. Although both formulas have been shown to be correct by BCFW recursion relations, our result is the first direct check using the regularized definition of the new formula.

Contents

1. Introduction

S-matrix program [1] is a program to study properties of quantum field theory based on some general principles, like the Lorentz invariance, Locality, Causality, Gauge symmetry as well as Analytic property. Because it does not use specific information like Lagrangian, result obtained by this method is quite general. Also exactly because its generality with very few assumptions, study along this line is very challenging.

One of the most important recent progresses in S-matrix program is the derivation of BCFW recursion relations in gauge theories [2, 3] and gravity [4], which relies only on basic analytic properties of tree amplitudes if there are no boundary contributions[1]. Furthermore, in [7], by assuming the applicability of BCFW recursion relations in gauge theories and gravity, many well-known (but difficult to prove) fundamental facts about S-matrix, such as non-Abelian structure for gauge theory and all matters couple to gravity with same coupling constant, have been re-derived from S-matrix viewpoint[2].

Based on these developments, non-trivial relations among tree-level color-ordered gauge theory amplitudes, including the recently proposed Bern-Carrasco-Johansson(BCJ) relations [9] (see also some applications [13]), have been proved using BCFW recursion relations in [10], which provided the first field-theoretical (S-matrix) proof of these relations[3]. Using similar ideas for gravity, new forms of Kawai-Lewellen-Tye(KLT) type relations [14] (for a good review, see [15]), which express gravity tree amplitudes as square of gauge theory amplitudes, have been found and proved in [16, 17, 18, 19].

[1] The boundary behavior is one important subject to study. In [5], background field method has been applied to the study. In [6], the situation with nonzero boundary contributions has also been discussed. It will be interesting to study the boundary behavior in the frame of S-matrix program.

[2] Gauge theory three-point amplitudes are uniquely determined by Poincare symmetry, in [8] it has been proved that, through BCFW recursion relations, any higher-point tree amplitudes can be consistently constructed if and only if there exists a non-Abelian gauge group.

[3] The BCJ relations have also been proved in string theory [11, 12].

There are two forms of KLT relations. The form with manifest S_{n-2} permutation symmetry is proposed and proved in [16]. It is mostly suitable for a BCFW(pure S-matrix) proof, but needs regularization to be well-defined. The most general expression of the minimally (manifest S_{n-3} permutation) symmetric form, is proposed and proved in [19], which has included the well-known ansatz for KLT relations conjectured in [20] as a special case. This S_{n-3} symmetric form is most natural from string perspective, as originally proposed and proved in string theory [14]. Both S_{n-2} and S_{n-3} symmetric forms have been generalized to $\mathcal{N}=8$ SUGRA case with similar S-matrix proofs in [18], which naturally produce new identities among $\mathcal{N}=4$ SYM amplitudes, including all 'flipped identities' for gluon amplitudes [17](see also [21]). Through string theory or BCFW recursion relation, the equivalence relation between S_{n-2} and S_{n-3} symmetric forms has been established. However, both methods are indirect, thus a direct algebraic manipulation is desired. As a major result of this note, in the second part we will show that there is a direct derivation from S_{n-2} symmetric form to the minimal, S_{n-3} symmetric form.

Although KLT relations give graviton amplitudes in terms of gluon amplitudes no matter what the helicity configuration is, there are not much explicit expressions available for graviton amplitudes unlike the case of gluon amplitudes. Among all helicity configurations, one of them is exceptional, i.e., the so-called MHV (maximally-helicity-violating) amplitudes. In Yang-Mills case, the famous Parke-Taylor formula [22] for gluon MHV tree amplitudes is astonishingly simple. On the other hand, the case for gravity amplitudes is much more complicated, even in the MHV sector. Various explicit formulas of MHV gravity amplitudes have been proposed [23, 20, 24, 25, 26, 27, 28], which fall into two categories: those with manifest S_{n-2} permutation symmetry, such as the formula given by Elvang-Freedman [26], and those with S_{n-3} symmetry, such as the original BGK formula [23] and the equivalent Mason-Skinner formula [27]. However, most of these formulas have been derived from approaches other than KLT relations, and it is non-trivial to show that they are equivalent to each other[29]. In the following, we will show that one particularly simple formula, the Mason-Skinner formula, directly follows from the S_{n-3} symmetric KLT relations, given the Parke-Taylor formula for gauge theory MHV amplitudes as the input. In addition, we will discuss the relation between Elvang-Freedman formula and the S_{n-2} symmetric form of KLT relations. As a byproduct, we will obtain an infinite number of new formulas for gravity MHV amplitudes. The equivalence of all these formulas are ensured by our derivation of S_{n-3} symmetric form from S_{n-2} symmetric form.

The outline of the note is the following. In section two we will derive Mason-Skinner formula for MHV gravity amplitudes from the recently proposed S_{n-3} permutation symmetric form of KLT relations. In section three, we will show the equivalence of S_{n-2} symmetric form and S_{n-3} symmetric form of KLT relations, and as an application, we derive from the S_{n-2} symmetric form an infinite number of new formulas for MHV gravity amplitudes, which are equivalent to BGK formula and Elvang-Freedman formula. In the Appendix we give another regularization procedure for S_{n-2} symmetric KLT formula.

2. From new KLT to Mason-Skinner MHV gravity amplitude

As mentioned in the introduction, although we have had general KLT relations and in principle all graviton amplitudes can be obtained through results of gluon amplitudes, so far most explicit formulas for general graviton amplitudes are constrained to MHV-graviton amplitudes[4]. All these expressions of MHV-graviton amplitudes are also very different and it takes efforts to show the equivalence among them[29]. One of these expressions is given by Mason and Skinner[5][27] as follows

$$\mathcal{M}_{MS}^{MHV} = (-)^{n-3} \sum_{P(2,\ldots,n-2)} \frac{A^{MHV}(1,2,\ldots,n)}{\langle 1|n-1\rangle \langle n-1|n\rangle \langle n|1\rangle} \prod_{k=2}^{n-2} \frac{[k|P_{k+1}+\cdots+P_{n-1}|n\rangle}{\langle k|n\rangle} , \tag{2.1}$$

where the sum is over all S_{n-3} permutations of labels $(2,...,n-2)$. In this section we will show that starting from general KLT formula and applying it to the MHV case, one can get the Mason and Skinner formula.

Since the Mason and Skinner formula is with the sum over S_{n-3} permutations, it is natural to start from following S_{n-3} permutation symmetric KLT formula [17, 19]

$$\mathcal{M}_n^{KLT} = (-)^{n+1} \sum_{\alpha,\beta\in S_{n-3}} A(1,\alpha,n-1,n)\mathcal{S}[\beta|\alpha]_{P_1}\tilde{A}(n,\beta,1,n-1) , \tag{2.2}$$

where the function $\mathcal{S}$ is defined as [16, 17, 19]

$$\mathcal{S}[i_1,...,i_k|j_1,j_2,...,j_k]_{P_1} = \prod_{t=1}^{k}(s_{i_t 1} + \sum_{q>t}^{k}\theta(i_t,i_q)s_{i_t i_q}) \tag{2.3}$$

with $\theta(i_t,i_q)$ to be zero when pair (i_t,i_q) has same ordering at both sets $\mathcal{I},\mathcal{J}$ and otherwise, to be one. Here $s_{ij} = (P_i+P_j)^2 = 2P_i\cdot P_j$.

Function $\mathcal{S}$ defined above has some properties which will be useful for our discussions. The first one is the reversed property

$$\mathcal{S}[i_1,...,i_k|j_1,j_2,...,j_k]_{P_1} = \mathcal{S}[j_k,...,j_1|i_k,..,i_1]_{P_1} . \tag{2.4}$$

The second one is about the sum over permutations. To illustrate this property, firstly we observe that

$$P_{ij}(S[\beta|\alpha]_{P_1}) = S[P_{ij}(\beta)|P_{ij}(\alpha)]_{P_1} , \tag{2.5}$$

where P_{ij} is the permutation of label i and label j while all other labels unchanged. Using this property we have

$$\sum_\beta S[\beta|P_{ij}(\alpha)] = \sum_\beta P_{ij}(S[P_{ij}(\beta)|\alpha]) = P_{ij}(\sum_\beta S[P_{ij}(\beta)|\alpha]) = P_{ij}(\sum_\beta S[\beta|\alpha]) , \tag{2.6}$$

[4]There are also some results for NMHV amplitudes and the general algorithm for $\mathcal{N}=8$ SUSY-Gravity[30, 31].

[5]We have written results in the QCD convention, which is different from the twistor convention by $[\,] \to -[\,]$.

where at the third equal sign we have used the factor that sum over all permutations $\sum_\beta$ is commutative with particular permutation P_{ij}. Then we have our second property

$$\sum_{\alpha\beta} F(\beta)S[\beta|\alpha]G(\alpha) = \sum_{P(2,\ldots,n-2)} (\sum_{\beta} F(\beta)S[\beta|2,\ldots,n-2]G(\{2,3,...,n-2\})) \ , \tag{2.7}$$

where $G(\alpha)$ is a general function. This property states that although corresponding terms in both sides of (2.7) are different under given permutations α, β, their summation is equivalent.

After stating these useful properties of function $\mathcal{S}$, let us continue our demonstration. Substituting (2.7) to formula (2.2) we get

$$\mathcal{M}_n^{KLT-MHV} = (-)^{n+1} \sum_{P(2,\ldots,n-2)} A^{MHV}(1,2,\ldots,n-1,n) \sum_{\beta} S[\beta|2,\ldots,n-2]\tilde{A}^{MHV}(n,\beta,1,n-1) \ . \tag{2.8}$$

After comparing (2.1) with (2.8), we see that we need to prove the following identity

$$\begin{aligned} &\frac{1}{\langle 1|n-1\rangle \langle n-1|n\rangle \langle n|1\rangle} \prod_{k=2}^{n-2} \frac{[k|P_{k+1}+\cdots+P_{n-1}|n\rangle}{\langle k|n\rangle} \\ &= \sum_{\beta} S[\beta|2,3,\ldots,n-2]\tilde{A}^{MHV}(n,\beta,1,n-1) \ . \end{aligned} \tag{2.9}$$

Note that label $(n-2)$ at the right hand part of function $\mathcal{S}$ is at the last position, and in this case we can divide permutations $\beta \in S_{n-3}$ into groups of permutations $\gamma \in S_{n-4}$ plus label $(n-2)$ inserted at all possible positions in sequence fixed by γ. Using this observation we can write down

$$\begin{aligned} &\sum_{\beta} \mathcal{S}[\beta|2,3,\ldots,n-2]_{P_1} A(n,\beta,1,n-1) \\ &= \sum_{\gamma\in P(2,\ldots,n-3)} \left(\sum_{\sigma\in OP(\{n-2\}\cup\{\gamma\})} \mathcal{S}[\sigma|2,\ldots,n-2]_{P_1} A(n,\sigma,1,n-1) \right) \\ &= \sum_{\gamma} \mathcal{S}[\gamma|2,\ldots,n-3]_{P_1} s_{n-2,n-1} A(n-2,n,\gamma,1,n-1) \\ &= s_{n-2,n-1} \sum_{\gamma} \mathcal{S}[\gamma|2,\ldots,n-3]_{P_1} A(n-2,n,\gamma,1,n-1) \ , \end{aligned} \tag{2.10}$$

where "OP" stands for "Ordered Permutation" and $OP(\{\alpha\}\cup\{\beta\})$ means all possible permutations which preserve elements' order of both sets. Here, because the set $\{n-2\}$ has only one element, $OP(\{n-2\}\cup\{\gamma\})$ just indicates all possible $(n-2)$ insertions into $\{\gamma\}$. Note also that at the second line, the function $\mathcal{S}$ has $(n-3)$ labels while at the third line, only $(n-4)$ labels are left. In the above derivation from the second line to the third line, we have used

$$\sum_{\sigma\in OP(\{n-2\}\cup\{\gamma\})} \mathcal{S}[\sigma|2,\ldots,n-2]_{P_1} A(n,\sigma,1,n-1) = \mathcal{S}[\gamma|2,\ldots,n-3]_{P_1} s_{n-2,n-1} A(n-2,n,\gamma,1,n-1) \ , \tag{2.11}$$

which directly follows from the level one BCJ relation(see [10]). As an example let us show the details of calculation of $n = 5$ case, which is

$$\begin{aligned}
&\sum_{\sigma\in OP(\{3\}\cup\{2\})} \mathcal{S}[\sigma|2,3]_{P_1} A(5,\sigma,1,4) \\
&= \mathcal{S}[2,3|2,3]_{P_1} A(5,2,3,1,4) + \mathcal{S}[3,2|2,3]_{P_1} A(5,3,2,1,4) \\
&= s_{21}\left(s_{31} A(5,2,3,1,4) + (s_{31}+s_{32}) A(5,3,2,1,4)\right) \\
&= \mathcal{S}[2|2]_{P_1} s_{34} A(3,5,2,1,4)\ ,
\end{aligned} \tag{2.12}$$

where in the fourth line, we have used $\mathcal{S}[2|2]_{P_1} = s_{21}$, the five-point level one BCJ relation

$$s_{31}A(5,2,3,1,4) + (s_{31}+s_{32})A(5,3,2,1,4) + (s_{31}+s_{32}+s_{35})A(3,5,2,1,4) = 0\ , \tag{2.13}$$

and the momentum conservation $s_{31} + s_{32} + s_{35} = -s_{34}$.

It is easy to see that, in the higher point case, $\sum_{\sigma\in OP(\{n-2\}\cup\{\gamma\})} \mathcal{S}[\sigma|2,\ldots,n-2]_{P_1}$ can always be written as a common factor $\mathcal{S}[\gamma|2,\ldots,n-3]_{P_1}$ multiplying the corresponding coefficients of level one BCJ relation, which we denote as $f_1(n-2)$ in the next section, since it is irrelevant with other elements except $(n-2)$.

We want to continue our simplification from the second line of (2.10) to the third line. However, it seems not possible to do any more simplification with the form given in (2.10) for general amplitudes. But when dealing with MHV amplitudes, there is "inverse soft factor"[32] which relates $(n-1)$-point MHV amplitude to n-point MHV amplitude as follows (it can be easily seen from Parke-Taylor formula [22])

$$A^{MHV}(n-1,n-2,n,\gamma,1) = \frac{\langle n-1|n\rangle}{\langle n-1|n-2\rangle\,\langle n-2|n\rangle} A^{MHV}(\widetilde{n-1},\tilde{n},\gamma,1,)\ , \tag{2.14}$$

where in order to preserve the momentum conservation, i.e., $P_{\widetilde{n-1}} + P_{\tilde{n}} = P_{n-1} + P_{n-2} + P_n$, spinor components have been modified as

$$\begin{aligned}
|\widetilde{n-1}] &= \frac{|P_{n-2}+P_{n-1}|n>}{\langle n-1|n\rangle}\ , \qquad |\widetilde{n-1}> = |n-1>\ , \\
|\tilde{n}] &= \frac{|P_{n-2}+P_n|n-1>}{\langle n|n-1\rangle}\ , \qquad |\tilde{n}> = |n>\ .
\end{aligned} \tag{2.15}$$

For (2.14) to be true we have assumed that the helicity of label $(n-2)$ is positive. This choice can always be made for graviton MHV amplitudes where we can fix, for example, label $1, n$ to be negative helicities. Also, only anti-spinor parts of momenta $P_{\widetilde{n-1}}, P_{\tilde{n}}$ have been changed while the spinor parts are untouched. This observation will be very useful for our later manipulation.

With this in mind we can continue our demonstration by substituting (2.14) into (2.10) and get

$$\begin{aligned}
&\sum_{\beta\in S_{n-3}} \mathcal{S}[\beta|2,3,\ldots,n-2]_{P_1} A_n^{MHV}(n,\beta,1,n-1) \\
&= s_{n-2,n-1}\frac{\langle n-1|n\rangle}{\langle n-1|n-2\rangle\,\langle n-2|n\rangle} \sum_{\gamma\in S_{n-4}} \mathcal{S}[\gamma|2,\ldots,n-3]_{P_1} A_{n-1}^{MHV}(\tilde{n},\gamma,1,\widetilde{n-1})\ ,
\end{aligned} \tag{2.16}$$

where summation in the second line is similar to the one in the first line except that the sum changes from S_{n-3} to S_{n-4}. Then we can iterate the procedure like the one did in (2.10) and yield

$$\begin{aligned}
&\sum_{\beta} S[\beta|2,3,\ldots,n-2]A^{MHV}(n,\beta,1,n-1)\\
&= s_{n-2,n-1}\frac{\langle n-1|n\rangle}{\langle n-1|n-2\rangle\langle n-2|n\rangle}s_{n-3,\widetilde{n-1}}\frac{\langle n-1|n\rangle}{\langle n-1|n-3\rangle\langle n-3|n\rangle}\times\\
&\qquad\sum_{\gamma'\in P(2,\ldots,n-4)}\mathcal{S}[\gamma'|2,\ldots,n-4]A^{MHV}(\tilde{\tilde{n}},\gamma',1,\widetilde{\widetilde{n-1}})\\
&=\cdots\\
&= S[2|2]A^{MHV}(\tilde{n}^{(n-4)},2,1,\widetilde{n-1}^{(n-4)})\prod_{k=3}^{n-2}s_{k,\widetilde{n-1}^{(n-2-k)}}\frac{\langle n-1|n\rangle}{\langle n-1|k\rangle\langle k|n\rangle}\ ,
\end{aligned}\tag{2.17}$$

where the notation $\tilde{n}^{(i)}$ means that there are i-th changing of momentum P_n. Using (2.15) it is easy to get the anti-spinor part of $\tilde{n}^{(i)}$

$$\begin{aligned}
|\widetilde{n-1}^{(i)}] &= \frac{|P_{n-1-i}+P_{\widetilde{n-1}^{(i-1)}}|n>}{\langle n-1|n\rangle} = \frac{|P_{n-1-i}+P_{n-i}+P_{\widetilde{n-1}^{(i-2)}}|n>}{\langle n-1|n\rangle}\\
&= \cdots = \frac{|P_{n-1-i}+P_{n-i}+\cdots+P_{n-2}+P_{n-1}|n>}{\langle n-1|n\rangle}\ .
\end{aligned}\tag{2.18}$$

Putting (2.18) back to (2.17) we get

$$\begin{aligned}
&S[2|2]A^{MHV}(\tilde{n}^{(n-4)},2,1,\widetilde{n-1}^{(n-4)})\prod_{k=3}^{n-2}s_{k,\widetilde{n-1}^{(n-2-k)}}\frac{\langle n-1|n\rangle}{\langle n-1|k\rangle\langle k|n\rangle}\\
&= \frac{1}{\langle 1|n-1\rangle\langle n-1|n\rangle\langle n|1\rangle}\frac{-[2|1|n\rangle}{\langle 2|n\rangle}\prod_{k=3}^{n-2}\frac{\langle n-1|n\rangle[k|\widetilde{n-1}^{(n-2-k)}]}{\langle k|n\rangle}\\
&= \frac{1}{\langle 1|n-1\rangle\langle n-1|n\rangle\langle n|1\rangle}\frac{-[2|1|n\rangle}{\langle 2|n\rangle}\prod_{k=3}^{n-2}\frac{[k|P_{k+1}+P_{k+2}+\cdots+P_{n-1}|n\rangle}{\langle k|n\rangle}\\
&= \frac{1}{\langle 1|n-1\rangle\langle n-1|n\rangle\langle n|1\rangle}\prod_{k=2}^{n-2}\frac{[k|P_{k+1}+P_{k+2}+\cdots+P_{n-1}|n\rangle}{\langle k|n\rangle}\ ,
\end{aligned}\tag{2.19}$$

where in the fourth line, we have rewritten $-[2|1|n\rangle$ as $[2|P_3+P_4+\cdots+P_{n-1}|n\rangle$ using momentum conservation. This finishes the proof of (2.9) .

Before ending this section, there is one application of above derivation we want to address. In [16, 17, 18, 21], new quadratic vanishing identities have been found and using them, one can reduce the independent helicity bases from $(n-3)!$ down further. For example, if we chose A to be non-MHV and $\widetilde{A}$ to be MHV, we have

$$0 = (-)^{n+1}\sum_{\alpha,\beta\in S_{n-3}}A^{non-MHV}(1,\alpha,n-1,n)\mathcal{S}[\beta|\alpha]_{P_1}\tilde{A}^{MHV}(n,\beta,1,n-1)\ .\tag{2.20}$$

Since identity (2.9) is true as long as $\widetilde{A}$ is MHV, we obtain immediately

$$0 = \sum_{\alpha \in S_{n-3}(2,..,n-2)} \frac{A^{non-MHV}(1,\{2,3,..,n-2\},n-1,n)}{\langle 1|n-1\rangle \langle n-1|n\rangle \langle n|1\rangle} \prod_{k=2}^{n-2} \frac{[k|P_{k+1}+\cdots+P_{n-1}|n\rangle}{\langle k|n\rangle} \tag{2.21}$$

for amplitudes A which are not MHV-amplitudes. This result has been presented in [18] where many other identities can be written down too.

3. From S_{n-2} KLT to S_{n-3} KLT

One important result of recent study of KLT relations is the manifest S_{n-2} permutation symmetric KLT formula presented in [16]

$$\mathcal{M}_n^{new} = (-1)^n \sum_{\gamma,\beta \in S_{n-2}} \frac{\tilde{A}_n(n,\gamma,1)\mathcal{S}[\gamma|\beta]_{P_1} A_n(1,\beta,n)}{s_{123\ldots(n-1)}} \ . \tag{3.1}$$

Formula (3.1) is not intuitive seeing from the familiar KLT relations presented in [20], even with the help of new discovered BCJ relations[9]. However, as shown in [19], this formula is the consistent requirement of the pure field understanding of S_{n-3} permutation symmetric KLT relation under the BCFW expansion and in fact, it is found by this way. Comparing to the formula given in [20], formula (3.1) is much easy to prove using BCFW recursion relations in field theory while its stringy derivation is still missing. Although formulas (2.2) and (3.1) are equivalent seen from BCFW recursion relations, in this section we will try to establish more direct relation between them.

3.1 The direct derivation

As emphasized in [16, 19], naively (3.1) seems to be ill-defined since $s_{123\ldots(n-1)}$ vanishes on-shell. However, there is a specific regularization under which (3.1) is a well-defined finite expression. The regularization is given by following off-shell continuation of momenta p_1 and p_n with an arbitrary momentum q[16, 19]

$$p_1 \to p_1 - xq, \quad p_n \to p_n + xq \ . \tag{3.2}$$

In order to have the on-shell condition for p_1, we need to impose $p_1 \cdot q = 0$ and $q^2 = 0$, while $q \cdot p_n \neq 0$. Thus we have $p_{\hat{1}}^2 = 0$ and $p_{\hat{n}}^2 = s_{\hat{1}23\ldots(n-1)} \neq 0$. Then a more accurate definition of (3.1) is the following limit

$$\mathcal{M}_n^{new} = (-1)^n \lim_{x\to 0} \sum_{\gamma,\beta \in S_{n-2}} \frac{\tilde{A}_n(\hat{n},\gamma,\hat{1}) S[\gamma|\beta]_{\widehat{P}_1} A_n(\hat{1},\beta,\hat{n})}{s_{\hat{1}23\ldots(n-1)}} \ , \tag{3.3}$$

where we have used "$\hat{}$" to remind us the off-shell regularization scheme. Now the denominator becomes

$$s_{\hat{1}23\ldots(n-1)} = p_{\hat{n}}^2 = (p_n + xq)^2 = x \cdot s_{nq} \neq 0 \ , \tag{3.4}$$

which means when taking the limit we only need to consider the linear coefficient of x in the numerator.

One important observation of (3.3) is that we only need to regularize one kind of these two amplitudes, because in the numerator either combination $\sum_\beta S[\gamma|\beta]A_n(1,\beta,n)$ or $\sum_\gamma \tilde{A}_n(n,\gamma,1)S[\gamma|\beta]$ vanishes due to the level one BCJ relation. If we denote, after regularization, one combination to be $f(x)$, the remaining amplitudes to be $g(x)$ and the denominator to be $h(x)$, then we have[6]

$$\begin{cases} \lim\limits_{x\to 0} f(x) = 0\ , \quad \lim\limits_{x\to 0} h(x) = 0\ , \quad \lim\limits_{x\to 0} \dfrac{f(x)}{h(x)} = \text{const}\ . \\ \lim\limits_{x\to 0} g(x) \neq 0 \end{cases}$$
$$\Longrightarrow \lim_{x\to 0}\frac{g(x)f(x)}{h(x)} = g(0)\cdot \lim_{x\to 0}\frac{f(x)}{h(x)} = \lim_{x\to 0} g(0)\frac{f(x)}{h(x)}\ , \tag{3.5}$$

which shows directly that only one kind of amplitudes is needed to be regularized. Without loss of generality we choose to regularize $A_n(1,\beta,n)$, which simplifies (3.3) to

$$\mathcal{M}_n^{new} = (-1)^n \lim_{x\to 0}\sum_{\gamma\beta}\frac{\tilde{A}_n(n,\gamma,1)S[\gamma|\beta]_{\widehat{P}_1}A_n(\widehat{1},\beta,\widehat{n})}{s_{\hat{1}23\ldots(n-1)}}\ . \tag{3.6}$$

We want to simplify (3.6) further. As we have seen in previous section, the last label in the sequence given by β, which denoted by β_{n-2}, possess a nice property. Under this consideration we regroup the two summations over γ,β as follows

$$\begin{aligned} &\sum_{\gamma,\beta\in S_{n-2}} \tilde{A}_n(n,\gamma,1)\mathcal{S}[\gamma|\beta]_{\widehat{P}_1}A_n(\hat{1},\beta,\hat{n}) \\ =& \sum_{\beta\in S_{n-2}} A_n(\hat{1},\beta,\hat{n}) \sum_{\gamma\in S_{n-2}} \tilde{A}_n(n,\gamma,1)\mathcal{S}[\gamma|\beta_1,...,\beta_{n-3},\beta_{n-2}]_{\widehat{P}_1} \\ =& \sum_{\beta} A_n(\hat{1},\beta,\hat{n}) \sum_{\gamma(\beta_{n-2})\in S_{n-3}}\left[\sum_{\sigma\in OP(\{\gamma(\beta_{n-2})\}\cup\{\beta_{n-2}\})} \tilde{A}_n(n,\sigma,1)\mathcal{S}[\sigma|\beta_1,...,\beta_{n-3},\beta_{n-2}]_{\widehat{P}_1}\right], \end{aligned} \tag{3.7}$$

where we have divided the permutation sum $\gamma\in S_{n-2}$ into the permutation sum $\gamma(\beta_{n-2})\in S_{n-3}$[7] plus all possible insertions of β_{n-2}. With the fixed β-ordering, we have

$$\mathcal{S}[\sigma|\beta_1,...,\beta_{n-3},\beta_{n-2}]_{\widehat{P}_1} = \mathcal{S}[\gamma(\beta_{n-2})|\beta_1,...,\beta_{n-3}]_{\widehat{P}_1} f_{\hat{1}}(\beta_{n-2})\ , \tag{3.8}$$

where $f_{\hat{1}}(\beta_{n-2})$ (mentioned before) is the kinematic factor provided by element β_{n-2}. In other words, the dependence of insertion positions of β_{n-2} is given completely by the factor $f_{\hat{1}}(\beta_{n-2})$. The dependence of deformed momentum $\hat{1}$ inside factor $f_{\hat{1}}(\beta_{n-2})$ is given by $s_{\beta_{n-2},\hat{1}} = s_{\beta_{n-2},1} - x s_{\beta_{n-2},q}$, thus we have

$$f_{\hat{1}}(\beta_{n-2}) = f_1(\beta_{n-2}) - x s_{\beta_{n-2},q}\ . \tag{3.9}$$

[6] We would like to thank T. Sondergaard for discussions on this point.

[7] $\gamma(\beta_{n-2})$ means the element β_{n-2} has been excluded.

As mentioned before, the key point is that

$$\sum_{\sigma\in OP(\{\gamma(\beta_{n-2})\}\cup\{\beta_{n-2}\})}\tilde{A}_n(n,\sigma,1)f_1(\beta_{n-2})=0 \tag{3.10}$$

by level-one BCJ relation, since $\widetilde{A}$ are un-deformed amplitudes. Putting all together we finally have

$$\begin{aligned}
&\sum_{\gamma,\beta\in S_{n-2}}\tilde{A}_n(n,\gamma,1)\mathcal{S}[\gamma|\beta]_{\widehat{P}_1}A_n(\hat{1},\beta,\hat{n})\\
&=\sum_{\beta}A_n(\hat{1},\beta,\hat{n})\sum_{\gamma(\beta_{n-2})\in S_{n-3}}(-xs_{\beta_{n-2},q})\mathcal{S}[\gamma(\beta_{n-2})|\beta_1,...,\beta_{n-3}]_{\widehat{P}_1}\left[\sum_{\sigma\in OP(\{\gamma(\beta_{n-2})\}\cup\{\beta_{n-2}\})}\tilde{A}_n(n,\sigma,1)\right]\\
&=\sum_{\beta}A_n(\hat{1},\beta,\hat{n})\sum_{\gamma(\beta_{n-2})\in S_{n-3}}(-\tilde{A}_n(n,\gamma(\beta_{n-2}),1,\beta_{n-2})(-xs_{\beta_{n-2},q})\mathcal{S}[\gamma(\beta_{n-2})|\beta_1,...,\beta_{n-3}]_{\widehat{P}_1}\ ,
\end{aligned} \tag{3.11}$$

where we have used $U(1)$-decoupling relation for label β_{n-2} in the third line, i.e.,

$$\sum_{\sigma\in OP(\{\gamma(\beta_{n-2})\}\cup\{\beta_{n-2}\})}\tilde{A}_n(n,\sigma,1)=-\tilde{A}_n(n,\gamma(\beta_{n-2}),1,\beta_{n-2})\ . \tag{3.12}$$

With expression (3.11) we can take the limit

$$\begin{aligned}
\mathcal{M}_n^{new}&=(-1)^n\lim_{x\to 0}\sum_{\gamma\beta}\frac{\tilde{A}_n(n,\gamma,1)S[\gamma|\beta]_{\widehat{P}_1}A_n(\hat{1},\beta,\hat{n})}{s_{\hat{1}23\ldots(n-1)}}\\
&=(-1)^n\lim_{x\to 0}\frac{\sum_\beta A_n(\hat{1},\beta,\hat{n})s_{\beta_{n-2}q}\sum_{\gamma(\beta_{n-2})\in S_{n-3}}\tilde{A}_n(n,\gamma(\beta_{n-2}),1,\beta_{n-2})\mathcal{S}[\gamma(\beta_{n-2})|\beta_1,...,\beta_{n-3}]_{\widehat{P}_1}}{s_{nq}}\\
&=(-1)^n\frac{\sum_\beta A_n(1,\beta,n)s_{\beta_{n-2}q}\sum_{\gamma(\beta_{n-2})\in S_{n-3}}\tilde{A}_n(n,\gamma(\beta_{n-2}),1,\beta_{n-2})\mathcal{S}[\gamma(\beta_{n-2})|\beta_1,...,\beta_{n-3}]_{P_1}}{s_{nq}}\ ,
\end{aligned} \tag{3.13}$$

where in the last step we have taken the $x\to 0$ limit so momenta p_1,p_n in A are the un-deformed ones. In order to continue further, we write the sum $\sum_{\beta\in S_{n-2}}=\sum_{\beta_{n-2}=2}^{n-1}\sum_{\beta(\beta_{n-2})\in S_{n-3}}$ and get

$$\mathcal{M}_n^{new}=-\sum_{\beta_{n-2}=2}^{n-1}\frac{s_{\beta_{n-2}q}}{s_{nq}}T_n(1,\beta_{n-2},n)\ , \tag{3.14}$$

with

$$\begin{aligned}
&T_n(1,\beta_{n-2},n)\\
&=(-)^{n+1}\sum_{\beta(\beta_{n-2}),\gamma(\beta_{n-2})\in S_{n-3}}A_n(1,\beta(\beta_{n-2}),\beta_{n-2},n)\mathcal{S}[\gamma(\beta_{n-2})|\beta(\beta_{n-2})]_{P_1}\tilde{A}_n(n,\gamma(\beta_{n-2}),1,\beta_{n-2})\ .
\end{aligned} \tag{3.15}$$

It is straightforward to see that $T_n(1, \beta_{n-2}, n)$ is nothing but the graviton amplitude expression given in (2.2) with labels $1, n, \beta_{n-2}$ fixed. Then if using the total symmetric property of graviton amplitudes, we obtain immediately

$$\mathcal{M}_n^{new} = -\sum_{\beta_{n-2}=2}^{n-1} \frac{s_{\beta_{n-2}q}}{s_{nq}} T_n(1, \beta_{n-2}, n) = -\mathcal{M}_n^{KLT} \sum_{\beta_{n-2}=2}^{n-1} \frac{s_{\beta_{n-2}}}{s_{nq}} = \mathcal{M}_n^{KLT} , \tag{3.16}$$

where in the last step we have used the momentum conservation and $s_{1q} = 0$.

There is one more thing we want to discuss before ending this part. In our proof, in order to show that the new KLT formula with manifest S_{n-2} permutation symmetry is equivalent to the old KLT formula with manifest S_{n-3} permutation symmetry, we have used the total symmetric property of old KLT formula, or at least the S_{n-2} permutation symmetry. This total symmetric property can be seen from string theory, however it is not so obvious from field theory. To show the property is true in field theory, one way is to do algebraic manipulations using BCJ relations. However, with a few examples, it can be seen that such calculations are very complicated with the increasing number of gravitons.

There is an indirect way to prove the total symmetric property of KLT relations. The key point is to adapt induction method through BCFW recursion relations. The three-point amplitudes are obviously total symmetric by Lorentz symmetry and spin. Since graviton amplitudes can be calculated by BCFW recursion relations, we can build up higher point amplitudes from lower point amplitudes, which have been assumed to be symmetric. Since all different KLT expressions give same physical quantity, they must be equivalent to each other, thus the total symmetric property is obtained. This idea has already been used in [19].

3.2 Application

One obvious consequence of our proof is that if we do not use the symmetry argument to pull out T in (3.14), we will have a new KLT formula with manifest S_{n-2} permutation symmetry like (3.1), but without the singular denominator. This formula depends on an arbitrary auxiliary momentum q as long as $q \cdot p_1 = 0$. Applying (2.9) to (3.14), with some manipulations we obtain

$$\mathcal{M}_n^{new-MHV} = \sum_{\beta \in S_{n-2}} \frac{\langle n|1\rangle \langle n-1|n-2\rangle\, s_{(n-1)q}}{\langle 1|n-1\rangle \langle n|n-2\rangle\, s_{nq}} F(1, \{2, .., n-1\}, n) , \tag{3.17}$$

where we have defined the following function

$$F(1, 2, .., n) = A(1, 2, .., n) \frac{\langle n|n-2\rangle}{\langle 1|n\rangle^2 \langle n|n-1\rangle \langle n-1|n-2\rangle} \prod_{s=2}^{n-2} \frac{\langle n|K_{(n-1)s}|s]}{\langle n|s\rangle} \tag{3.18}$$

with $K_{(n-1)s} = p_{n-1} + p_{n-2} + ... + p_s$. If we continue algebraic manipulation like one has done from (3.14) to (3.16) we obtain

$$\mathcal{M}_n^{BGK} = \sum_{\beta \in S_{n-3}} \frac{\langle 1|n\rangle \langle n-1|n-2\rangle}{\langle 1|n-1\rangle \langle n|n-2\rangle} F(1, \{2, .., n-2\}, n-1, n) , \tag{3.19}$$

which is nothing but the BGK expression [23] rewritten by Elvang and Freedman in [26]. Using the same function F, in [26] a manifest S_{n-2} permutation symmetric MHV amplitude is given by[8]

$$\mathcal{M}_n^{EF} = \sum_{\alpha\in S_{n-2}} F(1,\alpha\{2,3,...,n-1\},n) \ . \tag{3.20}$$

Thus it is interesting to discuss the relation between (3.17) and (3.20).

In order to do so, we can simplify (3.17) by taking $q = |1\rangle\,|q]$ so that $q\cdot p_1 = 0$, then we obtain

$$\mathcal{M}_n^{MHV} = -\sum_{\beta\in S_{n-2}} \frac{\langle n-1|n-2\rangle\,[n-1|q]}{\langle n|n-2\rangle\,[n|q]} F(1,\{2,..,n-1\},n) \ . \tag{3.21}$$

Formula (3.21) is different from (3.20) and (3.19), but it can be checked that all of them are equivalent to each other by BCFW recursion relations. A few examples may be useful to demonstrate the relation between (3.21) and (3.20). The case $n=3$ are simply $-\frac{\langle 1|2|q]}{\langle 1|3|q]}F(1,2,3) = F(1,2,3)$ by momentum conservation. For $n=4$ we have

$$-F(1,2,3,4)\frac{\langle 2|3|q]}{\langle 2|4|q]} - F(1,3,2,4)\frac{\langle 3|2|q]}{\langle 3|4|q]} \ .$$

We can take a special case that $q = k_2$, then the second term is zero and we obtain $-F(1,2,3,4)\frac{\langle 2|3|2]}{\langle 2|4|2]}$. The BGK formula (3.19) is $F(1,2,3,4)\frac{\langle 1|4\rangle\langle 3|2\rangle}{\langle 1|3\rangle\langle 4|2\rangle}$. In order to show above two results are consistent we check the following expression

$$\frac{\frac{\langle 2|3|2]}{\langle 2|4|2]}}{\frac{\langle 1|4\rangle\langle 3|2\rangle}{\langle 1|3\rangle\langle 4|2\rangle}} = \frac{\langle 1|4|2]}{\langle 1|3|2]} = -1 \ ,$$

which is true by momentum conservation.

We can learn from the different expressions (3.21) and (3.20) that the S_{n-2} permutation symmetric form has some redundancy, since the independent bases are $(n-3)!$ by BCJ relations.

Although in this note, we are not able to change form (3.21) to form (3.20) by direct algebraic manipulations, some identities about function F can be given by their equivalence. When using $\langle 1|n]$ BCFW-deformation

$$\lambda_1(z) = \lambda_1 + z\lambda_n, \quad \widetilde{\lambda}_n(z) = \widetilde{\lambda}_n - z\widetilde{\lambda}_1 \ , \tag{3.22}$$

$F(1,2,3,..,n)$ depends on z only through factor $\frac{1}{\langle 1|2\rangle}$ from $A(1,2,...,n)$, i.e., $F(1,2,...,n)$ contributes to the pole $s_{12}(z)$ only. Now let us consider the residue given by this pole from various MHV formulas. The formula (3.20) gives

$$\mathrm{Res}(\mathcal{M}_{s_{12}}^{EF}) = \frac{\langle 1|2\rangle}{\langle n|2\rangle} \sum_{\sigma\in S_{n-3}} F(1,2,\sigma(3,...n-1),n) \ , \tag{3.23}$$

[8]Using the bonus relation[29], it has been proved that (3.20) is equivalent to (3.19).

while the formula (3.21) gives

$$\mathrm{Res}(\mathcal{M}_{s_{12}}^{new-MHV}) = -\frac{\langle 1|2\rangle}{\langle n|2\rangle}\sum_{\beta\in S_{n-3}}\frac{\langle n-1|n-2\rangle\,[n-1|1]}{\langle n|n-2\rangle\,[n|1]}F(1,2,\{3,..,n-1\},n)\ , \tag{3.24}$$

where we have taken $|q] = |1]$. The BGK formula gives

$$\mathrm{Res}(\mathcal{M}_{s_{12}}^{BGK-1}) = \frac{\langle 1|2\rangle}{\langle n|2\rangle}\sum_{P(3,..,n-2)}\frac{\langle 2|n\rangle\,\langle n-1|n-2\rangle}{\langle 2|n-1\rangle\,\langle n|n-2\rangle}F(1,2,\{3,...,n-2\},n-1,n)\ , \tag{3.25}$$

and if we exchange $2\leftrightarrow(n-1)$ in BGK formula and take the residue, we obtain

$$\mathrm{Res}(\mathcal{M}_{s_{12}}^{BGK-2}) = \sum_{k=3}^{n-2}\frac{\langle 2|n-2\rangle\,\langle 1|k\rangle}{\langle k|2\rangle\,\langle n|n-2\rangle}\sum_{\sigma}F(1,k,\sigma,2,n)\ . \tag{3.26}$$

Since the residue is unique, above four expressions must be equal to each other. Note that each expression has $(n-3)!$ terms, thus we obtain relations between these $(n-3)!$ terms. This is consistent with the new discovered relations given in [16, 17, 18, 21].

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A. The symmetry of graviton amplitude

In section three, we have used the regularization procedure to show the equivalence of new KLT formula given in [16] with the ones given in [20, 17, 19]. There is also a direct, but much more complicated way to check this. The good point of this way is that we can see how the singular denominator $s_{12...(n-1)}$ appears in the algebraic manipulation, thus in this appendix we provide some details of this calculation. Before given explicit example, let us write down following procedure of calculations:

- Step one: Write down the expression (3.1).
- Step two: Choose a minimal basis for A_n and $\widetilde{A}_n$. These two basis (for A and for $\widetilde{A}$) can be different, but when the choice has been made, it must be kept in following calculations.
- Step three: Using BCJ-relations to express all remaining amplitudes of A-type and $\widetilde{A}$-type in terms of the chosen basis.

- Step four: Using momentum conservation ($s_{in} = -\sum_{j=1}^{n-1} s_{ij}$) to get rid of all p_n's that might be in the BCJ-relations. In other word, we have used $p_n = -\sum_{i=1}^{n-1} p_i$. But remember we can not use $s_{1n} = s_{23..(n-1)}$.

- Step five: Plugging the s_{in}-free BCJ-relations into the expression obtained from (3.1) and collecting corresponding coefficients of each basis. Every coefficient must have factor $s_{12\ldots n-1}$ in numerator, thus we can cancel the same singular factor in denominator.

- Step six: After the pole is canceled we can go on-shell again and use whatever known relations we want to reduce the expression into the familiar one, such as (2.2) etc.

The example we will demonstrate is the $n = 5$ case

$$(-)^5 M_5 = \widetilde{A}(5, \alpha(2,3,4), 1) \sum_{\alpha,\beta} \mathcal{S}[\alpha(2,3,4)|\beta(2,3,4)] A(1, \beta(2,3,4), 5) \tag{A.1}$$

Choosing $A(1,2,3,4,5)$ and $A(1,3,2,4,5)$ as a basis, other four orderings are given as following[9]

$$\begin{aligned}
A(1,3,4,2,5) &= \frac{s_{12}A(1,2,3,4,5) + (s_{12}+s_{32})A(1,3,2,4,5)}{s_{25}} \\
A(1,4,3,2,5) &= \frac{s_{12}(s_{24}+s_{45})A(1,2,3,4,5) - s_{13}s_{24}A(1,3,2,4,5)}{s_{14}s_{25}} \\
A(1,2,4,3,5) &= \frac{(s_{23}+s_{13})A(1,2,3,4,5) + s_{13}A(1,3,2,4,5)}{s_{35}} \\
A(1,4,2,3,5) &= \frac{-s_{12}s_{34}A(1,2,3,4,5) - s_{13}(s_{14}+s_{24})A(1,3,2,4,5)}{s_{14}s_{35}}
\end{aligned} \tag{A.2}$$

It is worth to observe that the first and third one are the level one BCJ relation, i.e., the denominator has only one s_{ij}, while the second and fourth one are level two (with two s_{ij} factors) BCJ relations[9] . For general n, this expansion needs to use up to level $(n-3)$ BCJ relations. Having the result (A.2), we can calculate various terms by our rule (do not forget to write, for example, $s_{35} = -s_{31} - s_{32} - s_{34}$). For example, with $\alpha(2,3,4) = (2,3,4)$ we have

$$\begin{aligned}
& A(1,2,3,4,5)s_{21}s_{31}s_{41} + A(1,2,4,3,5)s_{21}s_{41}(s_{31}+s_{43}) \\
& +A(1,3,2,4,5)s_{31}(s_{21}+s_{23})s_{41} + A(1,4,3,2,5)s_{41}(s_{31}+s_{34})(s_{21}+s_{23}+s_{24}) \\
& +A(1,3,4,2,5)s_{31}s_{41}(s_{21}+s_{23}+s_{24}) + A(1,4,2,3,5)s_{41}(s_{21}+s_{24})(s_{31}+s_{34}) \\
= & \frac{s_{1234}}{s_{35}}(s_{31}+s_{34})[-s_{12}s_{34}A(1,2,3,4,5) - s_{13}s_{24}A(1,3,2,4,5)]
\end{aligned}$$

[9]Here we call the order of BCJ relations by the number of denominator in formulas given in [9]. It is worth to notice that while the level one BCJ relation has been proved in [11, 12, 10], higher order BCJ relations have not had a general proof although one can explicit check it order by order recursively. It will be very interesting to have a general proof.

where the factor s_{1234} appears in numerator. Collecting all six permutations together and getting rid of s_{1234} we obtain

$$\begin{aligned}
& [-s_{12}s_{34}A(1,2,3,4,5)-s_{13}s_{24}A(1,3,2,4,5)]\frac{(s_{31}+s_{34})}{s_{35}}\widetilde{A}(2,3,4,1,5) \\
&+[-s_{12}s_{34}A(1,2,3,4,5)-s_{13}s_{24}A(1,3,2,4,5)]\frac{s_{13}}{s_{35}}\widetilde{A}(2,4,3,1,5) \\
&+[-s_{12}s_{34}A(1,2,3,4,5)-s_{13}s_{24}A(1,3,2,4,5)]\frac{(s_{21}+s_{24})}{s_{25}}\widetilde{A}(3,2,4,1,5) \\
&+[-s_{12}s_{34}A(1,2,3,4,5)-s_{13}s_{24}A(1,3,2,4,5)]\frac{s_{12}}{s_{25}}\widetilde{A}(3,4,2,1,5) \\
&+\left[-\frac{s_{12}s_{34}s_{13}}{s_{35}}A(1,2,3,4,5)-\frac{s_{13}s_{24}(s_{21}+s_{23})}{s_{25}}A(1,3,2,4,5)\right]\widetilde{A}(4,2,3,1,5) \\
&+\left[-\frac{s_{12}s_{34}(s_{13}+s_{32})}{s_{35}}A(1,2,3,4,5)-\frac{s_{13}s_{24}s_{31}}{s_{25}}A(1,3,2,4,5)\right]\widetilde{A}(4,3,2,1,5)
\end{aligned} \tag{A.3}$$

To continue, we add first four lines to get

$$[-s_{12}s_{34}A(1,2,3,4,5)-s_{13}s_{24}A(1,3,2,4,5)](\widetilde{A}(3,2,4,1,5)+\widetilde{A}(2,3,4,1,5)) \ , \tag{A.4}$$

and then add last two lines to get

$$-s_{13}s_{24}A(1,3,2,4,5)\widetilde{A}(2,4,3,1,5)-s_{12}s_{34}A(1,2,3,4,5)\widetilde{A}(3,4,2,1,5) \ . \tag{A.5}$$

Adding these two together we finally have

$$\begin{aligned}
& s_{13}s_{24}A(1,3,2,4,5)\widetilde{A}(2,4,1,3,5)+s_{12}s_{34}A(1,2,3,4,5)\widetilde{A}(3,4,1,2,5) \\
&= -s_{13}s_{24}A(1,3,2,4,5)\widetilde{A}(3,1,4,2,5)-s_{12}s_{34}A(1,2,3,4,5)\widetilde{A}(2,1,4,3,5)
\end{aligned} \tag{A.6}$$

which is the familiar KLT relations.

From this example, it can be seen that the direct method is very complicated because we need to use various BCJ relations up to level $(n-3)$ and to sum up various terms to obtain an overall factor $s_{12..(n-1)}$. After got rid of $s_{12...(n-1)}$ from the sum over A, we need to use BCJ relations again to sum over $\widetilde{A}$. Although case by case one can check, it is hard to observe the general patterns to give a rigorous proof, thus it is better to use our regularization method to give the proof as we did in section three.

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